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LETTER TO THE EDITOR

Occupation times distribution for Brownian motion on graphs

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Abstract
Considering a Brownian motion on a general graph, we study the joint law for the occupation times on all the bonds. In particular, we show that the Laplace transform of this distribution can be expressed as the ratio of two determinants. We give two formulations, with arc or vertex matrices, for this result and discuss a simple example.

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During the years 1930–40, Levy [1] developed detailed studies of the Brownian motion (BM), discovering many interesting properties. Among others, he got several arc-sine laws concerning the 1D BM (BM on an infinite line). For such a process, starting at $t = 0$ from the origin O and stopping at time t, we denote by T the time spent in the region $x > 0$. Levy established, for T, the probability law [2]

$$P(T < u) = \frac{2}{\pi} \arcsin \sqrt{\frac{u}{t}} \tag{1}$$

with the density

$$\mathcal{P}_t(T) = \frac{1}{\pi} \frac{1}{\sqrt{T(t - T)}}. \tag{2}$$

In particular, the (double) Laplace transform of $\mathcal{P}_t(T)$ is written as

$$\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\gamma t} \int_0^t \mathrm{d}T\, \mathcal{P}_t(T)\, \mathrm{e}^{-\xi T} \equiv \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\gamma t} \langle \mathrm{e}^{-\xi T} \rangle = \frac{1}{\sqrt{\gamma(\gamma + \xi)}}. \tag{3}$$

(From now on, $\langle \cdots \rangle$ stands for averaging over all Brownian curves starting from O.)

Since that time, Barlow and his collaborators [3] generalized this law in the following way.

Instead of a pure 1D BM, they considered (see figure 1) a Brownian particle starting from O and moving on a set of n semi-infinite lines originating from O. Moreover, each time the

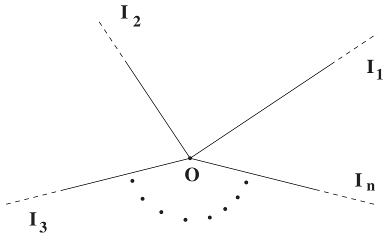

Figure 1. In [3], the Brownian particle is allowed to move on a set of n semi-infinite lines originating from O.

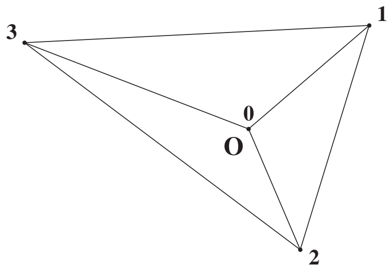

Figure 2. A graph with 4 vertices, 6 bonds and 12 arcs.

particle reaches O, it comes out with probability p_i in the direction I_i. In that context, the authors of [3] established the following result (T_i is the time spent on the ith leg):

$$\int_0^\infty \mathrm{d}t\ \mathrm{e}^{-\gamma t}\left\langle \mathrm{e}^{-\sum_{i=1}^n \xi_i T_i}\right\rangle = \frac{\sum_{i=1}^n \frac{p_i}{\sqrt{\gamma+\xi_i}}}{\sum_{i=1}^n p_i\sqrt{\gamma+\xi_i}}. \tag{4}$$

What is considered in [3] is a kind of special infinite graph. Our goal in this letter is to get an analogue of formula (4) but for a general graph. Remark that the interest of mathematicians [4] and, also, of physicists [5] in graphs is not new. In particular, the study of the spectral properties of the Laplacian operator on finite graphs, in view of physical applications (organic molecules, superconducting networks, weakly disordered systems, ...), began more than 50 years ago. Recently, graphs have also been used to study models of nonequilibrium statistical physics (see [6] and references therein).

So, let us start by considering such a general graph $\mathcal{G}$ made of V vertices, numbered from 0 to $V-1$, linked by B bonds of finite lengths. The coordination of vertex α is m_α $\left(\sum_{\alpha=0}^{V-1} m_\alpha = 2B\right)$.

On each bond $[\alpha\beta]$, of length $l_{\alpha\beta}$, we define the coordinate $x_{\alpha\beta}$ that runs from 0 (vertex α) to $l_{\alpha\beta}$ (vertex β). The set of coordinates $\{x_{\alpha\beta}\}$ is simply denoted x.

An arc $(\alpha\beta)$ is defined as the oriented bond from α to β. Each bond $[\alpha\beta]$ is, therefore, associated with two arcs $(\alpha\beta)$ and $(\beta\alpha)$. In the following, we will consider the following ordering of the $2B$ arcs: $(0\mu_1)(0\mu_2)\ldots(0\mu_{m_0})\ldots(\alpha\beta_1)\ldots(\alpha\beta_i)\ldots(\alpha\beta_{m_\alpha})\ldots$.

For example, for the graph of figure 2, we have $V=4$, $B=6$, and the sequence of the 12 ordered arcs is: (01)(02)(03)(10)(12)(13)(20)(21)(23)(30)(31)(32).

Now, let us suppose that a Brownian particle starts at $t=0$ from some vertex O (label 0) of the graph. At time t, this particle will reach some point that is left undetermined on the graph, and we denote as $T_{\alpha\beta}$ the time spent by this particle on the bond $[\alpha\beta]$.

Let $\mathcal{P}_t(\{T_{\alpha\beta}\})$ be the joint distribution of the occupation times $T_{\alpha\beta}$. In the following, we will study the Laplace transforms [7]:

$$\left\langle \mathrm{e}^{-\sum_{[\alpha\beta]}\xi_{\alpha\beta}T_{\alpha\beta}}\right\rangle \equiv \int \mathcal{P}_t(\{T_{\alpha\beta}\})\,\mathrm{e}^{-\sum_{[\alpha\beta]}\xi_{\alpha\beta}T_{\alpha\beta}}\prod_{[\alpha\beta]}\mathrm{d}T_{\alpha\beta} = \int_{\mathrm{Graph}}\mathrm{d}x\,\langle x|\,\mathrm{e}^{-tH}|0\rangle \tag{5}$$

$$\mathcal{L} \equiv \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-\gamma t}\left\langle \mathrm{e}^{-\sum_{[\alpha\beta]}\xi_{\alpha\beta}T_{\alpha\beta}}\right\rangle = \int_{\mathrm{Graph}}\mathrm{d}x\,\langle x|\frac{1}{H+\gamma}|0\rangle \equiv \int_{\mathrm{Graph}}\mathrm{d}x\,G(x). \tag{6}$$

On the bond $[\alpha\beta]$, H is the Hamiltonian $-\frac{1}{2}\Delta + \xi_{\alpha\beta}$ $\left(\Delta \equiv \frac{\mathrm{d}^2}{\mathrm{d}x_{\alpha\beta}^2}\right)$. Moreover, the behaviour of the resolvant $G(x)$ has to be specified in the neighbourhood of all the vertices.

To discuss this point, let us consider some vertex α with its nearest neighbours β_i, $i = 1, 2, \ldots, m_\alpha$, on the graph. Suppose that the Brownian particle reaches α. It will come out towards β_i with some arbitrary probability $p_{\alpha\beta_i}$. This implies $m_\alpha - 1$ equations to be satisfied by the resolvant:

$$\frac{G_{(\alpha\beta_1)}}{p_{\alpha\beta_1}} = \frac{G_{(\alpha\beta_2)}}{p_{\alpha\beta_2}} = \cdots = \frac{G_{(\alpha\beta_{m_\alpha})}}{p_{\alpha\beta_{m_\alpha}}} \tag{7}$$

where

$$G_{(\alpha\beta_i)} = \lim_{x_{\alpha\beta_i}\to 0} G(x_{\alpha\beta_i}). \tag{8}$$

Equation (7) can be established, for instance, by ‘discretizing’ the Brownian motion on each bond (i.e. by considering random walks with steps of lengths going to 0).

Remark that the resolvant will be continuous in vertex α only when the particle exits from α with the same probability in all directions.

Moreover, current conservation implies, if $\alpha \neq 0$

$$\sum_{i=1}^{m_\alpha} G'_{(\alpha\beta_i)} \equiv \sum_{i=1}^{m_\alpha}\frac{\mathrm{d}G}{\mathrm{d}x_{\alpha\beta_i}}\bigg|_{x_{\alpha\beta_i}=0} = 0. \tag{9}$$

On the other hand, on any bond $[0\mu_i]$ starting at vertex O, G must satisfy ($\gamma_{\alpha\beta} \equiv \gamma + \xi_{\alpha\beta}$)

$$\left(-\tfrac{1}{2}\Delta + \gamma_{0\mu_i}\right)G = \delta. \tag{10}$$

Spatial integration on an infinitesimal neighbourhood of O leads to

$$-\frac{1}{2}\sum_{i=1}^{m_0} G'_{(0\mu_i)} = 1. \tag{11}$$

Now, let us show that all the derivatives of G appearing in the above equations can be expressed in terms of the quantities $G_{(\alpha\beta)}$. On the link $[\alpha\beta]$, $G(x_{\alpha\beta})$ must satisfy

$$\left(-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\,x_{\alpha\beta}^2} + \gamma_{\alpha\beta}\right)G(x_{\alpha\beta}) = 0 \tag{12}$$

with the solution

$$G(x_{\alpha\beta}) = G_{(\alpha\beta)}\frac{\sinh\sqrt{2\gamma_{\alpha\beta}}(l_{\alpha\beta} - x_{\alpha\beta})}{\sinh\sqrt{2\gamma_{\alpha\beta}}l_{\alpha\beta}} + G_{(\beta\alpha)}\frac{\sinh\sqrt{2\gamma_{\alpha\beta}}x_{\alpha\beta}}{\sinh\sqrt{2\gamma_{\alpha\beta}}l_{\alpha\beta}}. \tag{13}$$

So, we deduce

$$G'_{(\alpha\beta)} = -c_{\beta\alpha}G_{(\alpha\beta)} + s_{\alpha\beta}G_{(\beta\alpha)} \tag{14}$$

with

$$c_{\alpha\beta} = \sqrt{2\gamma_{\alpha\beta}}\coth\sqrt{2\gamma_{\alpha\beta}}l_{\alpha\beta} = c_{\beta\alpha} \tag{15}$$

$$s_{\alpha\beta} = \frac{\sqrt{2\gamma_{\alpha\beta}}}{\sinh\sqrt{2\gamma_{\alpha\beta}}l_{\alpha\beta}} = s_{\beta\alpha}. \tag{16}$$

Equation (14) allows us to write equations (7), (9), (11) in a matrix form

$$MG = L \tag{17}$$

where G and L are two $(2B \times 1)$ vectors. The components of G are the quantities $G_{(\alpha\beta_i)}$ and for the components of L we have $L_j = 2\delta_{j0}$.

M is a $(2B \times 2B)$ arc matrix with the nonvanishing elements (α runs from 0 to $V - 1$)

$$M_{(\alpha\beta_1)(\alpha\beta_i)} = c_{\alpha\beta_i} \qquad M_{(\alpha\beta_1)(\beta_i\alpha)} = -s_{\alpha\beta_i} \qquad i = 1, \ldots, m_\alpha \tag{18}$$

$$M_{(\alpha\beta_i)(\alpha\beta_j)} = \frac{1}{p_{\alpha\beta_j}}(\delta_{ij} - \delta_{i+1,j}) \qquad i = 2, \ldots, m_\alpha - 1 \tag{19}$$

$$M_{(\alpha\beta_{m_\alpha})(\alpha\beta_j)} = \frac{1}{p_{\alpha\beta_j}}\left(\delta_{m_\alpha j} - \delta_{1j}\right). \tag{20}$$

Inverting the matrix M, it is now easy to get the quantities $G_{(\alpha\beta)}$.

Equations (6) and (13) lead to the analytic expression for the Laplace transform $\mathcal{L}$:

$$\mathcal{L} = \frac{\det M_1}{\det M}. \tag{21}$$

The matrix M_1 is the matrix M where the first line has been replaced by

$$(M_1)_{(0\mu_1)(\alpha\beta)} = \frac{c_{\alpha\beta} - s_{\alpha\beta}}{\gamma_{\alpha\beta}} \equiv a_{\alpha\beta}. \tag{22}$$

Equation (21) is what we call the arc matrix formulation. Let us now show that this result can be recast in terms of vertex matrices.

Multiplying the column $(\alpha\beta_j)$ of matrix M by $p_{\alpha\beta_j}$ and using standard properties of determinants, we readily get

$$\mathcal{L} = \frac{\det \mathcal{M}_1}{\det \mathcal{M}} \tag{23}$$

where $\mathcal{M}$ is now a $(V \times V)$ vertex matrix with elements (α, β run from 0 to $V - 1$)

$$\mathcal{M}_{\alpha\alpha} = \sum_{i=1}^{m_\alpha} p_{\alpha\beta_i} c_{\alpha\beta_i} \tag{24}$$

$$\mathcal{M}_{\alpha\beta} = -p_{\beta\alpha} s_{\alpha\beta} \qquad \text{if } [\alpha\beta] \text{ is a bond} \tag{25}$$

$$= 0 \qquad \text{otherwise}. \tag{26}$$

$\mathcal{M}_1$ is the same as matrix $\mathcal{M}$ except for the first line that is replaced by

$$(\mathcal{M}_1)_{0\alpha} = \sum_{i=1}^{m_\alpha} p_{\alpha\beta_i}\left(\frac{c_{\alpha\beta_i} - s_{\alpha\beta_i}}{\gamma_{\alpha\beta_i}}\right) \equiv \sum_{i=1}^{m_\alpha} p_{\alpha\beta_i} a_{\alpha\beta_i}. \tag{27}$$

It is worthwhile noting that, in general, vertex matrices are of a smaller size than arc matrices.

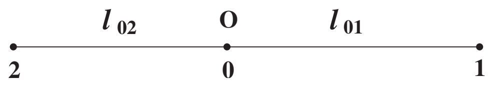

Figure 3. A graph with two finite legs starting from O.

To illustrate this work, let us consider the example of the graph of figure 3. For this graph we have, obviously, $p_{10} = p_{20} = 1$.

(i) *Arc formulation.* Matrices M and M_1 can be written as

$$M = \begin{pmatrix} c_{01} & c_{02} & -s_{01} & -s_{02} \\ -\frac{1}{p_{01}} & \frac{1}{p_{02}} & 0 & 0 \\ -s_{01} & 0 & c_{01} & 0 \\ 0 & -s_{02} & 0 & c_{02} \end{pmatrix} \qquad M_1 = \begin{pmatrix} a_{01} & a_{02} & a_{01} & a_{02} \\ -\frac{1}{p_{01}} & \frac{1}{p_{02}} & 0 & 0 \\ -s_{01} & 0 & c_{01} & 0 \\ 0 & -s_{02} & 0 & c_{02} \end{pmatrix}.$$

For $\mathcal{L}$, we get

$$\mathcal{L} = \frac{\sum_{i=1}^{2} \frac{p_{0i}}{\sqrt{\gamma_{0i}}} \tanh \sqrt{2\gamma_{0i}}\, l_{0i}}{\sum_{i=1}^{2} p_{0i} \sqrt{\gamma_{0i}} \tanh \sqrt{2\gamma_{0i}}\, l_{0i}} \tag{28}$$

$$\longrightarrow \frac{\sum_{i=1}^{2} \frac{p_{0i}}{\sqrt{\gamma_{0i}}}}{\sum_{i=1}^{2} p_{0i} \sqrt{\gamma_{0i}}} \qquad \text{when} \quad l_{0i} \to \infty. \tag{29}$$

The generalization to an n-leg graph is straightforward and leads to the result (4).

Now, let us briefly discuss equation (28).

When γ goes to infinity, (28) becomes equivalent to (29). The latest equation actually corresponds to the small time regime for the finite graph and the joint distribution $\mathcal{P}_t(\{T_{\alpha\beta}\})$ does not depend on the lengths of the links. This is because, in that regime, the particle does not have enough time to explore the whole graph; so, the legs appear to be infinite. In particular, for the mean occupation time $\langle T_{01} \rangle$ we get

$$\langle T_{01} \rangle \sim p_{01} t \qquad \text{when} \quad t \to 0^+. \tag{30}$$

On the other hand, the large time regime leads to

$$\langle T_{01} \rangle \sim \left(\frac{p_{01} l_{01}}{p_{01} l_{01} + p_{02} l_{02}} \right) t \qquad \text{when} \quad t \to \infty. \tag{31}$$

Finally, in the intermediate regime, $\langle T_{01} \rangle$ is, in general, no longer proportional to t. This is readily seen on its Laplace transform

$$\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\gamma t} \langle T_{01} \rangle = \frac{1}{\gamma^2} \frac{p_{01} \tanh \sqrt{2\gamma}\, l_{01}}{p_{01} \tanh \sqrt{2\gamma}\, l_{01} + p_{02} \tanh \sqrt{2\gamma}\, l_{02}}. \tag{32}$$

However, we observe that things become very simple when $l_{01} = l_{02}$: in that case, we get the same result as for an infinite graph, i.e. $\langle T_{01} \rangle = p_{01} t$ whatever t is.

(ii) *Vertex formulation.* Matrices $\mathcal{M}$ and $\mathcal{M}_1$ can be written as

$$\mathcal{M} = \begin{pmatrix} p_{01} c_{01} + p_{02} c_{02} & -s_{01} & -s_{02} \\ -p_{01} s_{01} & c_{01} & 0 \\ -p_{02} s_{02} & 0 & c_{02} \end{pmatrix} \qquad \mathcal{M}_1 = \begin{pmatrix} p_{01} a_{01} + p_{02} a_{02} & a_{01} & a_{02} \\ -p_{01} s_{01} & c_{01} & 0 \\ -p_{02} s_{02} & 0 & c_{02} \end{pmatrix}.$$

Obviously, they lead to the same result (28) for $\mathcal{L}$.

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